One-dimensional quantum models with correlated disorder versus classical oscillators with colored noise

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We perform an analytical study of the correspondence between a classical oscillator with frequency perturbed by a colored noise and the one-dimensional Anderson-type model with weak correlated diagonal disorder. It is rigorously shown that localization of electronic states in the quantum model corresponds to exponential divergence of nearby trajectories of the classical random oscillator. We discuss the relation between the localization length for the quantum model and the rate of energy growth for the stochastic oscillator. Finally, we examine the problem of electron transmission through a finite disordered lattice by considering the evolution of the classical oscillator.

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I. INTRODUCTION

This work serves the goal of establishing some quantitative links between two seemingly unrelated fields: quantum disordered models on the one hand and classical stochastic systems on the other. More precisely, we analyze the relations existing between a classical oscillator with frequency perturbed by a feeble noise and the one-dimensional (1D) Anderson-type model with a weak diagonal disorder. Our main interest is in *correlated* random potentials and, correspondingly, in *colored* noise for the stochastic oscillator.

Recently, the role of correlations in random potentials of quantum models has been the object of intense scrutiny. In particular, it was shown that specific long-range correlations in potentials may lead to the emergence of a continuum of extended states even in 1D lattices (see, e.g., [1] and [2] and references therein). In this paper we show that the phenomenon of Anderson localization has its counterpart in the energetic instability of a random oscillator. Specifically, the mobility edge generated in the 1D quantum models by long-range correlations is equivalent to the suppression of the energy growth of the stochastic oscillator due to temporal correlations of the frequency noise.

We use the correspondence between stochastic oscillators and disordered solid state models in order to study the transmission properties of finite lattices by making use of the dynamical analysis of an oscillator with noisy frequency. This approach allows us to put in a new perspective the problem of electronic transport in disordered lattices and also to gain new insight on the dynamics of random oscillators.

We observe that the analogy between stochastic oscillators and *continuous* disordered models of the Anderson kind has been investigated before (see, e.g., [3]). The novelty of the present work resides mainly in the following two features. In the first place, we analyze the correspondence between a stochastic oscillator and a *discrete* lattice, which is not so straightforward as the analogy of the former system with a continuous Anderson model. The second relevant aspect of this work is that we focus our attention on the physically new effects of long-range correlations.

This paper is organized as follows. In the following section we define the models that constitute the object of our study, and we make some general considerations on their analogies. In Sec. III we rigorously analyze the relation between the localization of electronic states for the Anderson model and the orbit instability of a random oscillator. In Sec. IV we discuss how correlations of the frequency noise can suppress the energy growth of the stochastic oscillator. The analogy between a random oscillator and a disordered chain is then used in Sec. V to study the electronic transmission through a finite disordered lattice. In Sec. V we also discuss the relation between energetic instability and orbit divergence for a random oscillator. Finally, Sec. VI is devoted to summarizing the conclusions.

II. DEFINITION OF THE MODELS

The Anderson model is defined by the discrete stationary Schrödinger equation

$$\psi_{n+1}+\psi_{n-1}+\varepsilon_n\psi_n=E\psi_n \tag{1}$$

where ψ_n is the amplitude of the wave function at the nth site of the lattice and disorder is introduced via the site energies ε_n which in the following are assumed to be *random correlated* variables. We do not restrict our considerations to a specific distribution for the random potential ε_n; we only suppose that it has zero average $\langle\varepsilon_n\rangle=0$ and that the binary correlator $\langle\varepsilon_n\varepsilon_{n+k}\rangle$ is a known function of the index k. We also assume that the correlator $\langle\varepsilon_n\varepsilon_{n+k}\rangle$ does not depend on n and that it is a decreasing function of k. In other words, we make the physically sensible assumptions that the random succession $\{\varepsilon_n\}$ is stationary, and that correlations decay with increasing distance. We restrict our analysis to the case of weak disorder, defined by the condition

$$\langle\varepsilon_n^2\rangle\ll 1.$$

In the preceding expressions the symbol $\langle\cdots\rangle$ stands for the average over a single disorder realization defined by the limit

$$\langle x_n \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n ;$$

we assume that this average is equivalent to the average over disorder realizations (ensemble average) for the succession $\{\varepsilon_n\}$.

It is known that the model (1) can be put into correspondence with the kicked oscillator defined by the Hamiltonian

$$H=\omega\left(\frac{x^2}{2}+\frac{p^2}{2}\right)+\frac{x^2}{2}\left(\sum_{n=-\infty}^{\infty} A_n \delta(t-nT)\right), \qquad (2)$$

which represents an oscillator whose momentum undergoes instantaneous variations of random intensity A_n at regular time intervals. The connection between the models (1) and (2) has been discussed before (see, e.g., [4]). Basically, the correspondence consists in the fact that, by integrating the Hamilton equations of motion of the oscillator (2) over the period between two successive kicks one gets the map

$$\begin{aligned} x_{n+1} &= x_n \cos(\omega T) + (p_n - A_n x_n)\sin(\omega T), \\ p_{n+1} &= -x_n \sin(\omega T) + (p_n - A_n x_n)\cos(\omega T), \end{aligned} \qquad (3)$$

where x_n and p_n stand for the position and momentum of the oscillator immediately before the nth kick. This map is equivalent to to the Schrödinger equation (1) which defines the Anderson model. Indeed, by eliminating the momentum from Eqs. (3), one gets the relation

$$x_{n+1}+x_{n-1}+A_n \sin(\omega T) x_n = 2x_n \cos(\omega T)$$

which coincides with the Schrödinger equation (1) provided that the position x_n of the oscillator at time $t=nT$ is identified with the electron amplitude ψ_n at the nth site and that the parameters of the kicked oscillator are related to those of the Anderson model by the identities

$$\varepsilon_n = A_n \sin(\omega T) \quad \text{and} \quad E = 2\cos(\omega T), \qquad (4)$$

The formal correspondence between the quantum model (1) and the kicked oscillator (2) raises the question of whether a similar analogy can link the Anderson model to a random oscillator whose frequency is perturbed by a continuous noise rather than by a succession of discontinuous and singular kicks as in model (2). In other words, one is led to infer the existence of close ties between the quantum model (1) and a stochastic oscillator defined by the Hamiltonian

$$H=\omega\left(\frac{x^2}{2}+\frac{p^2}{2}\right)+\frac{x^2}{2}\xi(t), \qquad (5)$$

where $\xi(t)$ is a continuous and stationary noise. Notice that these requirements on $\xi(t)$ set the random oscillator (5) and the kicked oscillator (2) in two different categories within the vast family of stochastic oscillators, since the succession of kicks in the model (2) is a nonstationary and strongly discontinuous random process. Consequently, the connection between the Anderson model (1) and the kicked oscillator (2) does not prove at all the equivalence of models (1) and (5) but only constitutes a hint that such a link may exist.

In our analysis of stochastic oscillators, we will focus on the Hamiltonians represented by Eq. (5), completing the definition of the model by further assuming that the noise $\xi(t)$ has zero average and that its binary correlator is a known function

$$\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t+\tau)\rangle = \chi(\tau). \qquad (6)$$

In Eq. (6) the symbol $\langle \cdots \rangle$ is used for the time average

$$\langle f(t) \rangle = \lim_{T_0\to\infty} \frac{1}{T_0}\int_0^{T_0} f(t)\,dt,$$

which is assumed to coincide with the ensemble average for the process $\xi(t)$. Notice that we do not restrict our consideration to the case of white noise, but we are instead interested in the general case of *colored* noise. Finally, we require that the noise $\xi(t)$ be weak; in other words, we assume that the fluctuations of the frequency around its average value are small.

Below we show that oscillators of the kind (5), with the above-mentioned noise features, are equivalent to the Anderson model (1) if two further conditions are met. First, the correlation function has to be of the form

$$\chi(\tau) = \frac{\langle A_n^2 \rangle}{T} \sum_{k=-\infty}^{+\infty} \zeta(k)\,\delta(\tau - kT), \qquad (7)$$

where the symbol $\zeta(k)$ stands for the normalized binary correlators

$$\zeta(k) = \frac{\langle A_{n+k} A_n \rangle}{\langle A_n^2 \rangle} \qquad (8)$$

of the random variables A_n specified by the second condition. Our second requirement is that the unperturbed frequency ω of the oscillator and the parameters A_n must be related to the parameters E and ε_n of the Anderson model through the identities (4).

Notice that the links established by these two conditions associate key features of the noise $\xi(t)$ to the corresponding properties of the random potential ε_n. Indeed, once the random variables ε_n and A_n are connected by the relation (4), the correlators (8) become identical to the normalized correlators of the potential ε_n. Therefore the spatial correlations of the disorder in the Anderson model are mirrored by temporal correlations for the noise $\xi(t)$. In the special case in which the disorder in the Anderson model is *uncorrelated* (i.e., $\langle \varepsilon_{n+k}\varepsilon_n \rangle = 0$ for $k \neq 0$), the noise for the random oscillator is *white* [i.e., $\langle \xi(t)\xi(t+\tau)\rangle \propto \delta(\tau)$]. One can also observe that the case of weak disorder in the Anderson model corresponds to that of weak noise for the random oscillator, since the condition $\langle \varepsilon_n^2 \rangle \ll 1$ entails the consequence that $\langle A_n^2 \rangle \ll 1$ (except that at the band edge, i.e., for $\omega T \to 0$, which is a special case where anomalies are expected to arise and will not be considered here).

Obviously, we must endow with a well-defined meaning the notion of ''equivalence'' used above to describe the connection between the Anderson model (1) and the random oscillator defined by Eqs. (5) and (7). We speak of the equivalence of the two models in the sense that the time evolution of the orbits of the random oscillator closely mirrors the spatial variation of the electronic states on the lattice. More precisely, the exponential divergence rate of nearby orbits turns out to be equal to the inverse localization length of the Anderson model.

The correspondence between the random oscillator (5) and the Anderson model (1) is to some extent surprising since the former is a *classical* system and is *continuous* in time whereas the latter model is *quantum* and *discrete* in space. It is therefore particularly interesting to notice how close the two systems turn out to be. To sum up, one of the main results of this paper is that the Anderson model with weak *correlated* disorder has a close analog in a random oscillator with frequency perturbed by a *colored* noise. This equivalence generalizes the result established in Ref. [5] where the Anderson model with uncorrelated disorder was linked to a random oscillator of the kind (5) with white noise.

III. THE LYAPUNOV EXPONENT

In the previous section we have described the analogy between the Anderson model (1) and the random oscillator (5) as being based on the correspondence between the electronic wave function of the former model and the space orbits of the latter system. To prove this analogy, we will compute the divergence rate of nearby trajectories of the random oscillator, i.e., its Lyapunov exponent and we will show that, when the conditions (4) and (7) are met, the Lyapunov exponent coincides with the inverse localisation length in the Anderson model, $\lambda = l_{\infty}^{-1}$. We define the Lyapunov exponent through the formula

$$\lambda = \lim_{T_0\to\infty}\lim_{\delta\to 0}\frac{1}{T_0}\,\frac{1}{\delta}\int_0^{T_0}\ln\frac{x(t+\delta)}{x(t)}\,dt. \tag{9}$$

To compute this expression it is convenient to introduce the polar coordinates defined through the standard relations $x = r\sin\theta$, $p = r\cos\theta$. This allows one to cast Eq. (9) in the form

$$\lambda = \lim_{T_0\to\infty}\frac{1}{T_0}\int_0^{T_0}\frac{\dot r}{r}\,dt.$$

To proceed further, we consider the dynamical equations for the random oscillator in polar coordinates

$$\dot\theta = \omega + \xi(t)\sin^2\theta, \tag{10}$$

$$\dot r = -\frac{1}{2}r\,\xi(t)\sin 2\theta. \tag{11}$$

Using the radial Eq. (11), the expression for the Lyapunov exponent can be finally put into the form

$$\lambda = -\lim_{T_0\to\infty}\frac{1}{2T_0}\int_0^{T_0}\xi(t)\sin[2\theta(t)]\,dt = -\frac{1}{2}\langle \xi(t)\sin[2\theta(t)]\rangle. \tag{12}$$

The problem of computing the Lyapunov exponent (9) is thus reduced to that of calculating the noise-angle correlator that appears in Eq. (12). This can be done in the following way, which is the extension to the continuum case of the procedure adopted in [1] for the discrete case. First, one introduces the noise-angle correlator defined by the relation

$$\gamma(\tau) = \langle \xi(t)\exp[2i\theta(t+\tau)]\rangle.$$

Starting from this definition, in the limit $\epsilon\to 0$ one has

$$\gamma(\tau+\epsilon) = \langle \xi(t)\exp[i2\theta(t+\tau)][1+2i\dot\theta(t+\tau)\epsilon]\rangle + o(\epsilon).$$

With use of the dynamical equation (10) one can further write

$$\gamma(\tau+\epsilon) = \gamma(\tau)(1+2i\omega\epsilon) + 2i\epsilon\langle \xi(t)\xi(t+\tau)\times\exp[2i\theta(t+\tau)]\sin^2\theta(t+\tau)\rangle + o(\epsilon).$$

In the limit of weak noise, one can factorize the correlator that appears in the right-hand side of the preceding equation and take the average over the angular variable using a flat distribution for θ. Indeed, when $\xi(t)\to 0$, Eq. (10) implies that $\dot\theta \simeq \omega$ so that, after a conveniently long time, one can expect the angular variable to take values uniformly distributed in the interval $[0,2\pi]$. As a consequence the noise-angle correlator must obey the relation

$$\gamma(\tau+\epsilon) = \gamma(\tau)(1+2i\omega\epsilon) - \frac{i}{2}\chi(\tau)\epsilon + o(\epsilon), \tag{13}$$

where $\chi(\tau)$ is the correlation function (or noise-noise correlator) defined by Eq. (6). On the other hand, a simple application of calculus rules leads to

$$\gamma(\tau+\epsilon) = \gamma(\tau) + \frac{d\gamma}{d\tau}(\tau)\epsilon + o(\epsilon). \tag{14}$$

Comparing Eqs. (13) and (14), one obtains the differential equation

$$\frac{d\gamma}{d\tau}(\tau) = 2i\omega\gamma(\tau) - \frac{i}{2}\chi(\tau),$$

whose solution [with the boundary condition $\lim_{\tau\to-\infty}\gamma(\tau) = 0$] gives the noise-angle correlator

$$\gamma(\tau) = -\frac{i}{2}\int_{-\infty}^{\tau}\chi(s)e^{2i\omega(\tau-s)}\,ds.$$

Using this result the Lyapunov exponent (12) can be finally written as

$$\lambda = \frac{1}{8}\int_{-\infty}^{+\infty} \langle \xi(t)\xi(t+\tau)\rangle \cos(2\omega\tau)\, d\tau, \tag{15}$$

which implies that the Lyapunov exponent for the stochastic oscillator (5) is proportional to the Fourier transform $\tilde{\chi}(2\omega)$ of the correlation function at twice the frequency of the unperturbed oscillator.

We are interested in the particular case in which the correlation function of the noise $\xi(t)$ takes the specific form (7), because we want to prove that in that case the Lyapunov exponent (15) coincides with the inverse localization length of the Anderson model (1). The substitution of the correlation function (7) in the general expression (15) gives

$$\lambda = \frac{\langle A_n^2 \rangle}{8T}\left[1 + 2\sum_{k=1}^{+\infty} \zeta(k)\cos(2\omega T k)\right].$$

Taking also into account the relations (4) between the parameters of the systems (1) and (5), one can finally write the Lyapunov exponent for the random oscillator in the form

$$\lambda = \frac{1}{T}\,\frac{\langle \varepsilon_n^2 \rangle}{8\sin^2(\omega T)}\,\varphi(\omega T),$$

$$\varphi(\omega T) = 1 + 2\sum_{k=1}^{+\infty} \zeta(k)\cos(2\omega T k). \tag{16}$$

This expression coincides with the one given in [1] for the localization length in the Anderson model with correlated disorder. The inverse localization length is given by the product of two factors, namely the Lyapunov exponent for the uncorrelated disorder case and the function $\varphi(\omega T)$, which describes the effect of disorder correlations (and which reduces to unity when correlations are absent). Formula (16) thus confirms the equivalence of the quantum Anderson model (1) with the classical oscillator (5) which had been inferred in Sec. II by the existence of a third system—the kicked oscillator (2)—which was somehow contiguous to both models (1) and (5). To sum up, formula (16) allows one to conclude that the Anderson model with the *correlated* disorder has a classical counterpart represented by a stochastic oscillator with the frequency perturbed by a *colored* noise. This conclusion generalizes the equivalence established in [5] between the Anderson model with *uncorrelated* disorder and an oscillator with the frequency perturbed by a *white* noise.

A remark is in order here: expression (16) for the inverse localization length of model (1) is correct for all energy values inside the unperturbed band *except* that at the band center i.e., for $\omega T = \pi/2$ where an anomaly arises and special methods are required for the analytical investigation (see, e.g., [6]). This anomaly is a resonance effect inherent in the discrete nature of the model (1) and cannot therefore be reproduced by the continuous system (5). Other anomalies appear in the Anderson model for the ''rational'' values of the energy (i.e., when $\omega T = \pi p/2q$ with p and q integer numbers), but they are effects of an order higher than the second [5] and need therefore not be considered here. In conclusion, apart from the exceptional case of the band center, the dynamical features of the models (1) and (5) do not differ to the second order of perturbation theory.

The equivalence of the models (1) and (5) can be examined also from a different point of view: that of the correspondence between discrete and continuous solid state systems. Indeed, the dynamical equation for the oscillator (5)

$$\ddot{x} + \omega\xi(t)x = -\omega^2 x \tag{17}$$

coincides, *mutatis mutandis*, with the stationary Schrödinger equation

$$-\psi'' + v(x)\psi = k^2\psi, \tag{18}$$

which describes the motion of a quantum particle of energy $E = k^2$ in a random potential $v(x) = -k\xi(x)$. Actually, expression (15) for the inverse localization length has long been known to solid state physicists (see, e.g., Ref. [7]) as the high-energy limit of the Lyapunov exponent for the continuous model (18); the same formula was later recovered in [3] using the correspondence of Eqs. (17) and (18) to deduce the Lyapunov exponent (15) from the analysis of the dynamics of the oscillator (17). In the present paper, rather than insisting further on the analogy between models (17) and (18), we prefer to draw a different conclusion, namely that the deduction of the inverse localization length (16) for the discrete Anderson model from expression (15) may be interpreted as the proof that the continuous model (18) can be put into one-to-one correspondence with the discrete lattice (1) *if, and only if* the correlation function of the random potential has the specific form (7). [Obviously, the transposition of results from one model to the other requires a proper change of the corresponding parameters with relations like (4); as a consequence of this swap, the mathematical correspondence of the two models does not imply an exact physical equivalence. Models (1) and (18), for instance, have different unperturbed energy spectra, defined by the respective dispersion relations $E = 2\cos k$ and $E = k^2$.]

IV. "MOBILITY EDGE" FOR A STOCHASTIC OSCILLATOR

In Ref. [1] the authors used formula (16) to investigate the problem of the mobility edge for the Anderson model (1). They showed that long-range correlations in the disorder can generate a continuum of extended electronic states and they found a way to construct sequences $\{\varepsilon_n\}$ of site energies giving rise to a Lyapunov exponent with a predefined dependence on the energy. In particular, using this recipe they were able to construct site potentials that generate a mobility edge even for the 1D lattice (1).

Here, we show how it is possible to solve the analogous problem for the random oscillator (5) taking formula (15) as a starting point. More precisely, we will show how to define a continuous noise $\xi(t)$ such that the corresponding Lyapunov exponent $\lambda(\omega)$ has a predefinite dependence on the frequency ω. Since the Lyapunov exponent determines the asymptotic behavior of the oscillator energy (we discuss this point more in detail in the next section), shaping the function $\lambda(\omega)$ through noise control enables one to determine the energetic behavior of the oscillator. In particular, if the noise $\xi(t)$ has the appropriate time correlations, the corresponding Lyapunov exponent can sharply drop from positive values to zero when the unperturbed frequency ω crosses a threshold value. In physical terms that means that the energetic growth of the oscillator is suppressed when the frequency reaches a critical value. The existence of a frequency threshold determining whether the oscillator is energetically stable or not is the physical counterpart of a mobility edge, which divides extended states from localized ones in the Anderson model. Thus, in spite of the current wisdom that frequency noise produces energetic instability (see, e.g., [8] and references therein), it turns out that time correlations of the noise may lead to a suppression of the energy growth. This conclusion follows directly from the known formula (15), but, to the best of our knowledge, this implication has not been discussed before in the literature.

A remark is in order here. Our analysis is based on a perturbative approach justified by the weak disorder assumption and our results for the Lyapunov exponent are correct to the second order of the expansion in the disorder strength. One should therefore keep in mind that our use of terms such as "mobility edge" or "suppression of the energy growth" is fully justified only within the limits of the second-order approximation.

To construct a noise $\xi(t)$ that gives rise to a defined Lyapunov exponent $\lambda(\omega)$, the starting point is the correlation function $\chi(\tau)$ that can be easily obtained by inverting formula (15)

$$\chi(\tau)=\frac{8}{\pi}\int_{-\infty}^{\infty}\lambda(\omega)e^{2i\omega\tau}d\omega.$$

Once the correlation function $\chi(\tau)$ is known, we can obtain a stochastic process $\xi(t)$ satisfying the conditions (6) by means of the convolution product

$$\xi(t)=(\beta*\eta)(t)=\int_{-\infty}^{+\infty}\beta(s)\,\eta(s+t)ds, \tag{19}$$

where the function $\beta(t)$ is related to the Fourier transform $\tilde{\chi}(\omega)$ of the noise correlation function through the formula

$$\beta(t)=\int_{-\infty}^{+\infty}\sqrt{\tilde{\chi}(\omega)}e^{i\omega t}\frac{d\omega}{2\pi}$$

and $\eta(t)$ is any stochastic process such that

$$\langle\eta(t)\rangle=0 \quad\text{and}\quad \langle\eta(t)\,\eta(t')\rangle=\delta(t-t'). \tag{20}$$

Formula (19) defines the family of noises corresponding to a specific form $\lambda(\omega)$ of the frequency-dependent Lyapunov exponent and constitutes the solution to the "inverse problem" [i.e., determination of a noise $\xi(t)$ that generates a predefined Lyapunov exponent].

As an example, we can consider the Lyapunov exponent

$$\lambda(\omega)=\begin{cases}1 & \text{if } |\omega|<1/2\\ 0 & \text{otherwise,}\end{cases} \tag{21}$$

whose frequency dependence implies that the random oscillator undergoes a sharp transition for $|\omega|=1/2$, passing from an energetically stable condition to an unstable one. Following the described procedure it is easy to see that the Lyapunov exponent (21) is generated by a noise of the form

$$\xi(t)=\frac{\sqrt{8}}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(s)}{s}\,\eta(s+t)ds,$$

with $\eta(t)$ being any random process with the statistical properties (20).

At this point, it is opportune to stress that the mathematical identity of Eqs. (17) and (18) implies that all features of the random oscillator (5) are shared by the solid state model (18). Therefore the mathematical results of this section not only imply that noise correlations can make the random oscillator (5) stable, they also represent a recipe to construct a random potential generating a mobility edge for the model (18).

V. TRANSMISSION THROUGH A FINITE DISORDERED LATTICE

We are now in the position to see how the analogy between the quantum model (1) and the random oscillator (5) can be used not only to compute the localization length in the Anderson model but also to deal with problems both more challenging and of greater physical interest, such as the study of the transmission properties of a finite disordered lattice. In this section we show how the random oscillator formalism allows us to tackle this problem and how it is possible to obtain expressions for the transmission coefficient as a function both of the sample length and of the inverse localization length (15).

More specifically, let us consider the case of a 1D disordered lattice of L sites sandwiched between two semi-infinite perfect leads. Mathematically the problem is defined by the Schrödinger equation (1), where the site energies ε_n are now equal to zero for $n<1$ and $n>L$, while for $1\leqslant n\leqslant L$ they are assumed to be correlated random variables. In [9] it was shown that the transmission coefficient T_L through the L-sites segment can be expressed in terms of the classical map (3) as

$$T_L=\frac{4}{2+r_1^2(L)+r_2^2(L)}, \tag{22}$$

where $r_1(L)$ and $r_2(L)$ represent the radii at the Lth step of the map trajectories starting from the phase-space points $P_1=(x_0=1,\ p_0=0)$ and $P_2=(x_0=0,\ p_0=1)$, respectively. An analogous formula was given in [7] for continuous models like the one defined by Eq. (18).

Formula (22) constitutes the bridge that makes it possible to link the transmission properties of a finite disordered lattice to the time evolution of the energy r^2 of the stochastic oscillator (5). Taking this formula as a starting point, one can analytically study the transport properties of a finite random lattice in two distinct cases: the ballistic regime, when the width of the barrier is much less that the localization length for the infinite lattice, and the localized regime, when the vice versa is true. The two cases are, respectively, identified by the conditions $L \ll l_\infty$ and $l_\infty \ll L$, where we use the symbol $l_\infty = \lambda^{-1}$ to denote the inverse of the Lyapunov exponent (16) and we are assuming that the lattice step is unitary, so that we can refer to L both as the number of sites of the disordered lattice and as the length of the barrier. We will evaluate the transmission properties first in the ballistic and then in the localized regime.

Before proceeding to the discussion of the two cases, we observe that our use of the continuous model (5) makes the results of this section valid for both continuous models like Eq. (18) and for the discrete lattice (1). The same formulas apply to both cases, with the localization length l_∞ taking the forms (15) or (16), depending on whether the formulas refer to the continuous or the discrete model. We also note that the results of this section were obtained long ago for continuous models (see, e.g., [7] and references therein); what is new here is their application to the discrete case and the approach used in their derivation, which sets the mathematical results in a different physical perspective.

A. The ballistic regime

In the ballistic regime, i.e., when $L \ll l_\infty$, one has $r_{1,2}(L) \simeq 1$ and expression (22) can be written in the form

$$T_L = 1 + \frac{2 - r_1^2(L) - r_2^2(L)}{4} + \cdots . \tag{23}$$

Another quantity of physical interest is the resistance of the finite disordered lattice, which is here defined as the inverse of the transmission coefficient

$$R_L = T_L^{-1} = \frac{2 + r_1^2(L) + r_2^2(L)}{4}. \tag{24}$$

A glance at expressions (23) and (24) reveals that, in order to obtain the *average* value of these physical quantities, one has to compute the average of the squared radii $r_1^2(L)$ and $r_2^2(L)$ over different disorder realizations. To achieve this goal, one can rely on the method developed by Van Kampen to study random oscillators and other stochastic models (see [8] and [10]). Van Kampen's approach is based on the construction of a dynamical equation for the average moments of the position and momentum of the random oscillator. For the second moments one has

$$\frac{d}{dt}\begin{pmatrix} \langle x^2 \rangle \\ \langle p^2 \rangle \\ \langle px \rangle \end{pmatrix} = \mathbf{A} \begin{pmatrix} \langle x^2 \rangle \\ \langle p^2 \rangle \\ \langle px \rangle \end{pmatrix}, \tag{25}$$

where the evolution matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 2\omega \\ \epsilon_1 + \epsilon_2 & -\epsilon_1 + \epsilon_2 & -2\omega \\ -\omega + \epsilon_3 & \omega & -\epsilon_1 + \epsilon_2 \end{pmatrix} \tag{26}$$

with

$$\epsilon_1 = \int_0^\infty \chi(\tau) d\tau,$$

$$\epsilon_2 = \int_0^\infty \chi(\tau) \cos(2\omega\tau) d\tau,$$

$$\epsilon_3 = \int_0^\infty \chi(\tau) \sin(2\omega\tau) d\tau.$$

For the general case of colored noise, Eq. (25) is correct up to order $O(\epsilon) = O(\xi^2)$; for the special case of white noise, however, it turns out to be *exact*.

One can extract substantial information from Eq. (25); in particular, it is possible to obtain the behavior of the average squared radii $r_1^2(t)$ and $r_2^2(t)$ for $t \to 0$

$$\langle r_1^2(t) \rangle = 1 + (\epsilon_1 + \epsilon_2)t + o(t^2),$$

$$\langle r_2^2(t) \rangle = 1 + (-\epsilon_1 + \epsilon_2)t + o(t^2).$$

As a consequence one has

$$\left\langle \frac{r_1^2(t) + r_2^2(t) - 2}{4} \right\rangle = \frac{1}{2}\epsilon_2 t + o(t^2). \tag{27}$$

Notice that these equations are correct up to order $O(t^2)$, so that it is meaningful to retain the distinction between the parameter ϵ_1 and ϵ_2. Using the result (27) one arrives at the following expressions for the average transmission coefficient and resistance:

$$\langle T_L \rangle = 1 - 2\frac{L}{l_\infty} + O\left(\left(\frac{L}{l_\infty} \right)^2 \right)$$

and

$$\langle R_L \rangle = 1 + 2\frac{L}{l_\infty} + o\left(\left(\frac{L}{l_\infty} \right)^2 \right).$$

These formulas show that in the ballistic regime the averages of both the transmissivity and the resistance are linear functions of the thickness L of the disordered layer. In addition, the average resistance coincides with the inverse of the average transmissivity

$$\langle T_L^{-1} \rangle \simeq \langle T_L \rangle^{-1}.$$

B. The localized regime

In the localized regime the disordered lattice extends over several localization lengths: $L \gg l_\infty$. In this case, to evaluate the average value of the transmission coefficient (22) it is convenient to determine the probability distribution for the random variable r. We observe that for $L \gg l_\infty$ the radius increases exponentially; moreover, one has

$$r_1(L) \simeq r_2(L) \simeq r(L), \tag{28}$$

with probability equal to one, regardless of the initial condition. As a consequence we can drop the subscripts 1 and 2 and write the transmission coefficient in the simplified form

$$\langle T_L \rangle \simeq \left\langle \frac{2}{1+r^2(L)} \right\rangle. \tag{29}$$

From the mathematical point of view, the problem of computing the average (29) can be better handled by introducing the logarithmic variable $z = \ln r$. The dynamics of the random oscillator (5) is then determined by the equations

$$\begin{aligned} \dot{z} &= -\frac{1}{2}\xi(t)\sin 2\theta, \\ \dot{\theta} &= \omega + \xi(t)\sin^2\theta. \end{aligned} \tag{30}$$

System (30) belongs to the class of stochastic differential equations of the form

$$\dot{u}_i = F_i^{(0)}(\mathbf{u}) + \alpha F_i^{(1)}(\mathbf{u},t), \tag{31}$$

where $F_i^{(0)}(\mathbf{u})$ represents a sure function of $\mathbf{u}$ perturbed by a stochastic function $\alpha F_i^{(1)}(\mathbf{u},\mathbf{t})$ with $\alpha \ll 1$. Indeed, one can reduce the system (30) to the form (31) by defining the vectors of Eq. (31) as

$$\mathbf{u} = \begin{pmatrix} z \\ \theta \end{pmatrix}, \quad \mathbf{F}^{(0)} = \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \quad \mathbf{F}^{(1)} = \begin{pmatrix} -\dfrac{1}{2\alpha}\xi(t)\sin(2\theta) \\ \dfrac{1}{\alpha}\xi(t)\sin^2\theta \end{pmatrix}$$

with $\alpha = \sqrt{\langle \xi^2(t) \rangle}$. It is known that a stochastic differential equation of the form (31) can be associated with a partial differential equation whose solution $P(\mathbf{u},t)$ represents the probability distribution for the random variable $\mathbf{u}$ [8]. This partial differential equation can be written as

$$\begin{aligned} \frac{\partial P}{\partial t} = \sum_i \frac{\partial}{\partial u_i} \Bigg\{ & -[F_i^{(0)} P(\mathbf{u},t)] \\ & + \alpha^2 \sum_j \int_0^\infty \left\langle F_i^{(1)}(\mathbf{u},t) \frac{d(u^{-\tau})}{d(u)} \frac{\partial}{\partial u_j^{-\tau}} F_j^{(1)}(\mathbf{u}^{-\tau}, t-\tau) \right\rangle \\ & \times \frac{d(u)}{d(u^{-\tau})} P(\mathbf{u},t) d\tau \Bigg\} + o(\alpha^2), \end{aligned} \tag{32}$$

where $\mathbf{u}^t$ stands for the flow defined by the deterministic equation $\dot{\mathbf{u}} = \mathbf{F}^{(0)}(\mathbf{u})$, $d(u^{-\tau})/d(u)$ is the Jacobian of the transformation $\mathbf{u} \to \mathbf{u}^{-\tau}$, and the symbol $o(\alpha^2)$ represents the omitted terms of order higher than the second in the perturbative parameter α. Thus, in the case of weak stochasticity ($\alpha \ll 1$), one can describe the dynamical behavior of the system (31) with an approximate equation of the Fokker-Planck kind.

In the present case, the approximate Fokker-Planck equation (32) associated with the dynamical system (30) takes the form

$$\begin{aligned} \frac{\partial P}{\partial t}(\theta,z,t) = & -\omega \frac{\partial P}{\partial \theta} + \frac{1}{4}\sin(2\theta)\frac{\partial}{\partial \theta} \\ & \times \left\{ [-\epsilon_1 + \epsilon_2 \cos(2\theta) + \epsilon_3 \sin(2\theta)] \frac{\partial P}{\partial z} \right\} \\ & + \frac{1}{2}\frac{\partial}{\partial \theta} \Bigg\{ \sin^2(\theta)[\epsilon_3 \cos(2\theta) - \epsilon_2 \sin(2\theta)] \frac{\partial P}{\partial z} \\ & + \sin^2(\theta) \frac{\partial}{\partial \theta} \{ [\epsilon_1 - \epsilon_2 \cos(2\theta) \\ & - \epsilon_3 \sin(2\theta)] P \} \Bigg\} \\ & + \frac{1}{4}\sin(2\theta)[\epsilon_2 \sin(2\theta) - \epsilon_3 \cos(2\theta)] \frac{\partial^2 P}{\partial z^2}. \end{aligned} \tag{33}$$

We remark that this equation is correct to the second order in $\xi(t)$ in the general case of colored noise; in the special case when the noise $\xi(t)$ is *white*, however, it can be shown that Eq. (33) becomes *exact*.

Once we dispose of the Fokker-Planck equation(33) for the general distribution $P(z,\theta,t)$, we can consider that, in order to evaluate the average of the transmission coefficient (22), we actually need only the probability distribution for the radial variable r (or for the equivalent logarithmic variable z). Therefore, we do not have to solve Eq. (33) in all its generality and we can instead consider the restricted Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^{2\pi} & P(\theta,z,t) d\theta \\ & = \frac{1}{8}\int_0^{2\pi} \{ [1-\cos(4\theta)]\epsilon_2 - \sin(4\theta)\epsilon_3 \} \frac{\partial^2 P}{\partial z^2} d\theta \\ & + \frac{1}{4}\int_0^{2\pi} \{ 2\epsilon_1 \cos(2\theta) \\ & - \epsilon_2 [1+\cos(4\theta)] - \epsilon_3 \sin(4\theta) \} \frac{\partial P}{\partial z} d\theta \end{aligned}$$

obtained by integrating Eq. (33) over the redundant angular variable. To proceed further we assume that, after a short-lived transient, the probability distribution takes the form

$$P(\theta,z,t) \simeq \frac{1}{2\pi} P(z,t). \tag{34}$$

This assumption can be justified on the grounds that, for weak noise, the dynamics of the angular variable is approximately ruled by the equation $\dot{\theta}\simeq\omega$. This implies that, after a sufficiently long time (of the order of some periods $2\pi/\omega$), the angular variable will have swept the whole interval $[0:2\pi]$ in an almost uniform way. That makes it reasonable to suppose that, for times $t\gg 2\pi/\omega$, the angular distribution is flat (excluding of course the exceptional case when $\omega\simeq 0$, i.e., when the energy value lies in a neighborhood of the band edge).

As a consequence of the hypothesis (34), one eventually gets the reduced Fokker-Planck equation for the z variable

$$\frac{\partial P}{\partial z}(z,t)=\lambda\left[-\frac{\partial P}{\partial t}(z,t)+\frac{\partial^2 P}{\partial z^2}(z,t)\right], \tag{35}$$

where λ is the Lyapunov exponent (16). Equation (35) has the form of a heat equation with a constant drift; its solution is therefore

$$P(z,t)=\frac{1}{\sqrt{2\pi\lambda t}}\exp\left[-\frac{(z-\lambda t)^2}{2\lambda t}\right]. \tag{36}$$

This solution satisfies the initial condition $P(z,t=0)=\delta(z)$, i.e., we have assumed that at time $t=0$ one has $r=1$, as is the case for the initial conditions P_1 and P_2. The initial condition, however, is somewhat arbitrary, since Eq. (35) is correct only for times $t\gg 2\pi/\omega$.

Knowledge of the distribution (36) makes it possible to compute the average transmission coefficient in the localized regime. Using probability (36) we can actually evaluate expression (29) and thus obtain

$$\langle T_L\rangle=\int_{-\infty}^{+\infty}\frac{2}{1+\exp(2z)}P(z,L)dz\simeq\sqrt{\frac{\pi l_\infty}{2L}}\exp\left(-\frac{L}{2l_\infty}\right). \tag{37}$$

As a result, in the limit $L\to\infty$ one has

$$-\frac{1}{L}\ln\langle T_L\rangle=\frac{\lambda}{2}. \tag{38}$$

Formulas (37) and (38) show that in the localized regime the transmission coefficient decreases exponentially with the width of the disordered lattice and they provide the correct rate of exponential decay. It must be pointed out, however, that expression (37) fails to reproduce the exact preexponential factor, which actually scales as $(l_\infty/L)^{3/2}$ (for approximation-free results see [7] and references therein). This partial shortcoming must be attributed to the two approximations made in the derivation of formula (37), i.e., (i) assumption (28) that allows the substitution of the exact expression (22) for the transmission coefficient with the simplified form (29) and (ii) hypothesis (34) about the angular dependency of the probability distribution $P(\theta,z,t)$. Both assumptions are admittedly incorrect for very short times, i.e., for distances L which are small on the length scale defined by l_∞. Thus we are led to the conclusion that in formula (37) the exponential factor is determined by the long-time behavior of the random oscillator (which is correctly described in our approach), while the preexponential factor is strongly influenced by the short-time dynamics of the oscillator. It is interesting to notice that an incorrect preexponential factor proportional to $(l_\infty/L)^{1/2}$ was also obtained in [7] studying a continuous solid-state model with a different approach. In that study, however, the physical meaning of the adopted simplifying hypotheses was not so transparent as in the present case, where the analogy between models (1) and (5) makes it possible to gain an intuitive comprehension of the mathematical approximations.

Beside allowing one to compute the average of the transmission coefficient, the probability distribution (36) makes it possible to determine the average value of other physical quantities which are relevant for a thorough description of the transport properties of a disordered lattice. The logarithm of the transmission coefficient and the resistance (24) are standard choices for the complete analysis of the conductance problem.

The interest for the logarithm of the transmission coefficient stems from the fact that, unlike the transmission coefficient itself, the logarithm $\ln T_L$ is a *self-averaging* variable and therefore a physically more sound parameter for the definition of the transport features of the disordered finite lattice (see, e.g., [7]). In the present framework, the average of the logarithmic transmissivity can be computed as follows. First, we observe again that in the localized regime condition (28) is fulfilled for almost every realization of the disorder so that we can write

$$\langle\ln T_L\rangle\simeq\left\langle\ln\frac{2}{1+r^2(L)}\right\rangle.$$

This expression can be put in the equivalent form

$$\langle\ln T_L\rangle=-\langle\ln(r^2)\rangle+\ln(2)-\left\langle\ln\left(1+\frac{1}{r^2}\right)\right\rangle. \tag{39}$$

We now observe that for every $x>0$ the logarithm satisfies the relations $0<\ln(1+x)<x$; hence the last term on the right-hand side (rhs) of the preceding equation must obey

$$0<\left\langle\ln\left(1+\frac{1}{r^2}\right)\right\rangle<\left\langle\frac{1}{r^2}\right\rangle=1, \tag{40}$$

where we have made use of distribution (36) to evaluate the average of $1/r^2$. Relations (39) and (40) imply that in the limit $L\to\infty$ one has

$$-\frac{1}{L}\langle\ln T_L\rangle=\frac{2}{L}\langle\ln r(L)\rangle.$$

Substituting in the rhs of this equation the average value of the variable $z=\ln r$ one finally obtains

$$\langle\ln T_L\rangle=-2\frac{L}{l_\infty}$$

which shows that the average logarithm of the transmission coefficient decreases linearly with the lattice width in the localized regime.

A third quantity that represents a meaningful statistical characteristic of the disordered lattice is given by the inverse of the transmission coefficient, i.e., by the resistance (24). As we did in the previous cases, we rely on the condition (28) to write the resistance in the form

$$R_L \simeq \frac{1}{2}[1+r^2(L)]. \tag{41}$$

Starting from this expression and making use of distribution (36) we obtain

$$\langle R_L \rangle \sim \exp\left(\frac{4L}{l_\infty}\right). \tag{42}$$

This expression shows that the average value of the resistance increases exponentially so that the resistance has a multiplicative rather that additive behavior as a function of the length of the disordered lattice. This conclusion obviously ceases to be valid in the special case in which long-range correlations of the random potential make the localization length l_∞ diverge: in this case the disordered lattice becomes transparent. We underline that, using the recipe given in Ref. [1] for the Anderson model—or the prescriptions of Sec. IV for the continuous model (18)—it is possible to define a random potential such that the corresponding Lyapunov exponent is zero in certain frequency intervals and positive elsewhere. As a consequence, the disordered lattice generated by such a potential will be transparent for electrons with the appropriate energies and opaque otherwise. This opens the possibility of projecting efficient electronic filters and agrees with the recent experimental findings discussed in [2].

As a further consideration, we observe that Eqs. (37) and (42) show that in the localized regime the inverse of the average transmission coefficient does *not* coincide with the average of the resistance

$$\langle T_L^{-1} \rangle \neq \langle T_L \rangle^{-1}$$

in contrast to the ballistic regime case.

At this point we wish to remark that the interest of expression (42) goes beyond the definition of the transport properties of a finite disordered lattice. This is so because the resistance R_L is strictly related to the energy r^2 of the random oscillator (5), as clearly shown by Eq. (41). The exponential increase of the average resistance can therefore be reinterpretated as energetic instability of the random oscillator (5) on long time scales and formula (42) can be rewritten in the alternative form

$$\langle r^2(t) \rangle \propto \exp(\gamma_E t) \quad \text{for} \quad t \gg 1,$$

with

$$\gamma_E = 4\lambda, \tag{43}$$

where λ is the Lyapunov exponent (15). This result shows that the energy of the random oscillator grows exponentially at large times (unless one has $\lambda=0$) and that the rate γ_E of this exponential increase is equal to *four* times the Lyapunov exponent. We could have computed the energy growth rate also with a different approach, taking Van Kampen's equation (25) as a starting point. In fact, Eq. (25) determines the time evolution of the second moments of the position and momentum of the random oscillator; it is therefore possible to obtain the result (43) by determining the eigenvalue of the evolution matrix (26) with the largest real part.

We would like to note that, in some papers devoted to stochastic classical systems, the rate of orbit divergence (Lyapunov exponent) is assumed to be a factor *two* less than the energy growth. The confusion probably stems from (and is equivalent to) the incorrect assumption that for large times the average of the logarithm of the energy and the logarithm of the average energy coincide, whereas the real relation is

$$\frac{1}{t}\langle \ln r^2(t) \rangle = \frac{1}{2t}\ln\langle r^2(t) \rangle,$$

valid in the limit $t \to \infty$.

As a last remark, we point out another consequence of the correspondence between the resistance of a disordered lattice and the energy of the stochastic oscillator (5). It is well known that in the localized regime the resistance R_L is a *non-self-averaged* quantity, since the relative fluctuations of this quantity do not disappear in the macroscopic limit. Indeed, if we employ the average value (42) of the resistance and use the distribution (36) to compute the average of the square of the resistance (41), we obtain that the root-mean-square deviation of the resistance behaves like

$$\delta R_L = (\langle R_L^2 \rangle \langle R_L \rangle^{-2} - 1)^{1/2} \propto \exp(2L/l_\infty), \tag{44}$$

i.e., it grows exponentially with the length of the random lattice. This result is well known to solid-state physicists, but it may be of some interest to reformulate it in terms of the dynamics of the stochastic oscillator (5). Then we can express the meaning of the result (44) by saying that the energy of the stochastic oscillator (5) is a quantity whose asymptotic value can fluctuate wildly from one noise realization to another. Relative fluctuations do not vanish at long times; this means that the concept of non-self-averaging quantity can find useful applications also in the field of stochastic classical systems.

VI. CONCLUSIONS

The first part of this paper is devoted to a thorough discussion of the analogies existing between the Anderson model (1) with diagonal correlated disorder and the stochastic oscillator (5) with frequency perturbed by a colored noise with correlation function (7). Our analysis shows that the two systems are equivalent in the sense that there exists a close correspondence between electronic states on one hand and space trajectories on the other. Quantitatively, this correspondence manifests itself in the identity of the inverse localization length for electronic states with the exponential rate of divergence of nearby oscillator trajectories. It is remarkable that this correspondence holds in spite of the fact that the Anderson model is quantum and discrete (in space) whereas the random oscillator is classical and continuous (in time). The analogy between the models (1) and (5) has been already investigated in [5] for the basic case of uncorrelated noise and disorder; however, the present work extends the previous conclusions to the case of *correlated disorder*.

In the second part of the work we discuss some implications of the parallelism between the models (1) and (5). In the first place, we translate the concept of ''mobility edge'' from the field of solid state physics to that of stochastic systems, showing how time correlations of the frequency noise may produce energetic stability for the random oscillator (5). We demonstrate that the knowledge of the oscillator dynamics on a finite time scale can be useful to gain insight about the transport properties of finite disordered lattices. This allows us to derive important results on electronic transmission in a simple and physically transparent way. Using the analogy the other way round, we can also deduce statistical properties of the energy of the stochastic oscillator from the knowledge of statistical features of the resistance of disordered wires. In passing we also clarify the relation between energetic and orbit instabilities for the random oscillator (5).

In conclusion, we believe that the bridge built among the fields of solid-state disordered systems and classical stochastic models represents a useful way to study the properties of both classes of systems. The present paper can be considered as an illustration of how this dual approach works, allowing one to solve old problems by putting them in a new perspective.

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